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Line Arrays: Theory and Applications

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ABSTRACT

Line arrays of loudspeakers are often employed to provide increased directivity, generally in the vertical plane. For improved performance, contemporary line arrays employ specially designed loudspeaker elements to provide a nearly continuous line source. However, even these may have "imperfections" relative to a perfect line source. This paper provides mathematical models to evaluate the directivity response of line sources and to quantify the effects of certain imperfections.

INTRODUCTION

Line arrays of loudspeakers are used by sound system designers to obtain a narrow directivity response, generally in the vertical plane. This narrow response provides increased gain that is useful in venues requiring long throw or improved direct/reverberant ratios. In the past, these line arrays were constructed out of closely stacked direct radiators or horns and worked quite well where the wavelength was large compared to the length (height) of the array. At shorter wavelengths the array elements cast their individual directional characteristics into the response of the array, resulting in deviations from the expected response of a "perfect" line array. Recently, certain loudspeaker systems have been designed specifically for use in line arrays. These produce a directivity response much closer to that of a perfect line source than earlier designs. Nonetheless, they are not perfect either. This paper provides a mathematical analysis of certain of these discrepancies.

Nearly all analyses of line arrays reference several seminal works from the late 1920's to the 1960's. This paper is no exception. We will begin with a derivation of the general form of the directivity function of a continuous line source and discuss important assumptions made to express it in closed form. The resulting directivity function provides the mathematical basis for several important topics related to line arrays including the quarter-power angle, the off-axis lobe/null structure, the distance to the far field and the on-axis pressure response. Each of these is discussed in the following sections.

In practice, even with specially designed loudspeaker systems, large line arrays are not perfectly continuous line sources at all frequencies. They have gaps between the boxes which are essentially non-radiating portions of the line. Also, certain radiating elements in the systems may produce radial wave fronts instead of pure, flat ribbons. These effects are particularly important at high frequency, where the deviations from a perfect line source are a material fraction of the wavelength. This paper closes with an analysis of how these imperfections affect the directivity response.

LINE ARRAYS – CONTINUOUS MODEL

The first step in deriving the directivity function of a line array is to develop an expression for the radiated pressure. A line source[a] can be modeled as a continuum of infinitely small line segments

[a] Line *array* and line *source* are used interchangeably. Line arrays generally refer to an array of acoustical sources such as loudspeakers arranged in a line. A line source is a mathematical representation of a large number of infinitely small radiating segments along a line.

distributed along a line [1]. The acoustic pressure radiated from a line source is

$$pressure = \int_0^l \frac{A(x)}{r(x)} e^{-j(kr(x)+\phi(x))} dx$$

where l is the length of the line source, $A(x)$ is the amplitude function along the line, k is the wave number, $\phi(x)$ is the phase function along the line, and $r(x)$ is the distance from any segment along the line to the point of observation **P**.

The evaluation of this expression is greatly simplified if we assume that the observation point **P** is a large distance away. That is, that the distance is much greater than the length of the array and the distance to **P** from any two segments along the line are approximately equal. This allows us to bring the $1/r(x)$ term in front of the integral since

$$\frac{1}{r(x)} \approx \frac{1}{r(0)} \approx \frac{1}{r(l)} \approx \frac{1}{r}.$$

Conversely, the $r(x)$ term in the exponential must be dealt with. This is because the relatively small distance differences to **P** from any two segments are not small compared to a wavelength. Figure 1 shows that $r(x)$ in the exponent can be expressed as

$$r(x) = x\sin(\alpha)$$

where α is the angle between a line normal to the axis of the source and a line from the source to **P**.

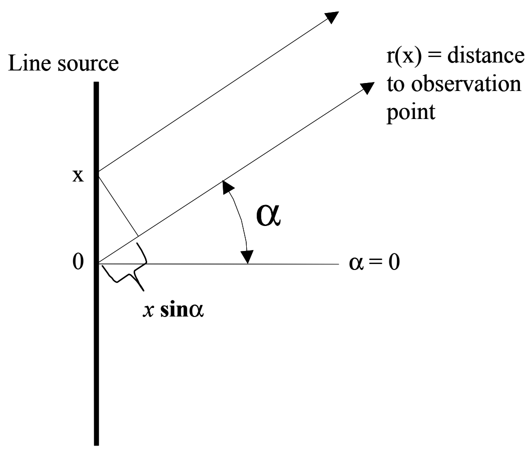

Figure 1: Geometric construction of relative distance function.

Substituting, the far field pressure at angle α of a continuous line source is

$$pressure(\alpha) = \frac{1}{r}\int_0^l A(x) e^{-j(kx\sin\alpha+\phi(x))} dx$$

The directivity function $R(\alpha)$ of a line source is defined as the magnitude of the pressure at angle α over the magnitude of the maximum pressure that can be obtained. That is,

$$R(\alpha) = \frac{|pressure|}{|pressure_{\max}|}.$$

The maximum radiated pressure is obtained when all segments along the line radiate in phase, i.e. the exponential function equals unity[b]. It is given as:

$$pressure_{\max} = \frac{1}{r}\int_0^l A(x)dx.$$

The general form of the directivity function $R(\alpha)$ of a line source is therefore

$$R(\alpha) = \frac{\left|\int_0^l A(x) e^{-j(kx\sin\alpha+\phi(x))} dx\right|}{\left|\int_0^l A(x)dx\right|} \tag{1}$$

UNIFORM LINE ARRAYS

Equation 1 is the general form of the line source directivity function at large distances. It is valid for any amplitude and phase functions along the length. A *uniform* line source has constant amplitude and phase along its length, that is, $A(x) = A$ and $\phi(x) = 0$. Substituting into Equation 1 yields

$$R(\alpha) = \frac{1}{l}\left|\int_0^l e^{-jkx\sin\alpha} dx\right|.$$

Solving this integral and applying Euler's identities, this becomes

$$R(\alpha) = \frac{\sin(\frac{kl}{2}\sin\alpha)}{\frac{kl}{2}\sin\alpha}$$

or, as more commonly expressed [2]

$$R(\alpha) = \frac{\sin(\frac{\pi l}{\lambda}\sin\alpha)}{\frac{\pi l}{\lambda}\sin\alpha}. \tag{2}$$

Figure 2 shows polar response curves[c] of a uniform amplitude and phase line array for various ratios of array length and wavelength. The polar response is wide at low ratios of l/λ. As the ratio increases the directivity pattern narrows and exhibits nulls and lobes.

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[b] Note that the maximum pressure at any given distance and frequency may never actually be obtained.

[c] A polar response curve is the directivity function expressed in decibels and plotted on a polar chart. The on-axis pressure is used as the reference pressure, i.e.

$$\text{Polar response} = 20Log\frac{R(\alpha)}{R(0)}.$$

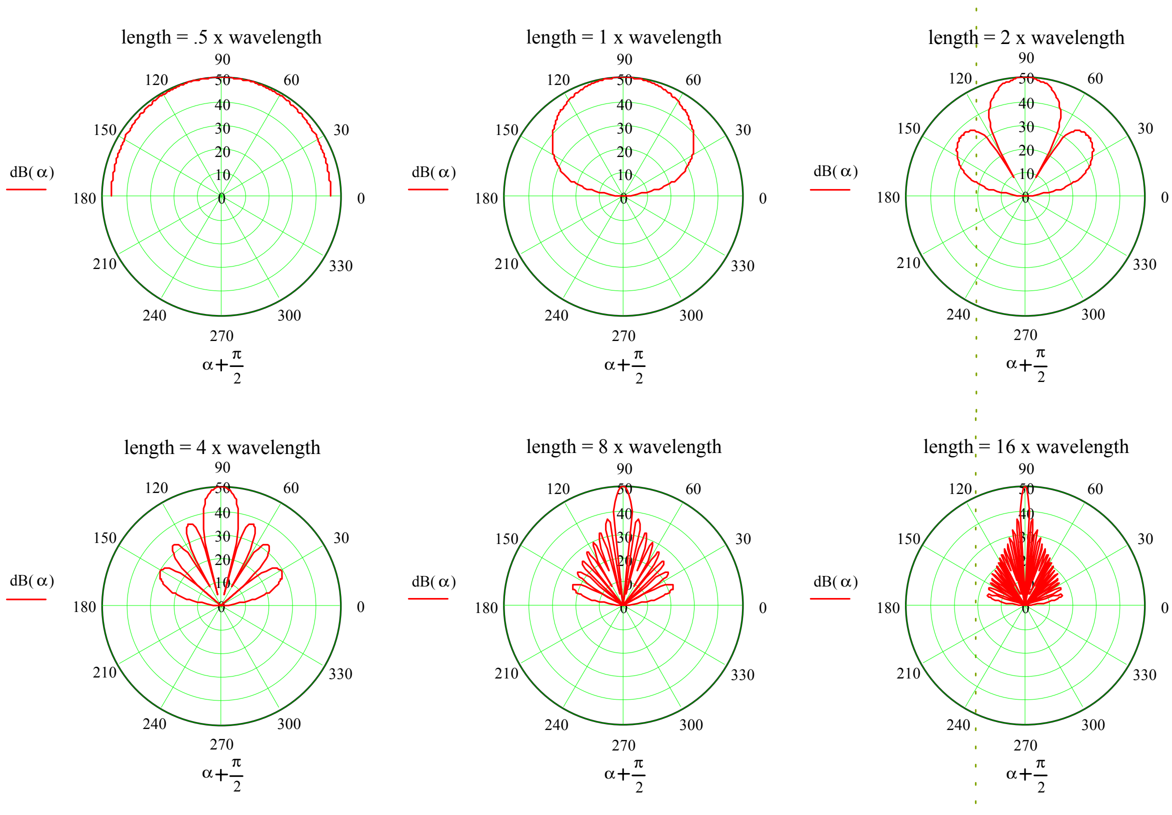

Figure 2: Polar response curves of a uniform line source.

LOBES AND NULLS OF UNIFORM LINE ARRAYS

Figure 2 shows that at long wavelengths ($\lambda > l$) the polar response of a line array is fairly omni-directional. At shorter wavelengths, lobes and nulls are obtained in the polar response curves. The position and magnitude of these are easily calculated.

The far field directivity function of a uniform line array has the generic form $\sin(u)/u$. On axis, that is $u = 0$, this expression must be evaluated using L'Hospital's rule, taking derivatives of the numerator and denominator. This yields

$$\lim \frac{\sin u}{u} \to \cos(0) = 1$$

showing that there will always be a lobe on-axis. Nulls are obtained when $\sin(u)/u$ goes to zero. This occurs where the argument u reaches (non-zero) multiples of π and can be expressed as

$$\frac{\pi l}{\lambda}\sin(\alpha) = m\pi$$

where m is an integer. Therefore, nulls are obtained at

$$|\sin(\alpha)| = m\frac{\lambda}{l} \quad \text{where } m = 1, 2, 3...$$

Lobes of the directivity function are found in-between the nulls, that is, at

$$|\sin(\alpha)| = \frac{3\lambda}{2l}, \frac{5\lambda}{2l}, \frac{7\lambda}{21}, \ldots.$$

This can be written as

$$|\sin(\alpha)| = \frac{(m+\frac{1}{2})\lambda}{l} \quad \text{where } m = 1, 2, 3...$$

Since the amplitude decreases inversely with u, the pressure amplitude of the m^{th} lobe is

$$A_m = \left|\frac{\cos(m\pi)}{m\pi + \frac{\pi}{2}}\right| \quad \text{where } m = 1, 2, 3...$$

QUARTER POWER ANGLE OF A UNIFORM LINE ARRAY

Often, a sound system designer is interested to know the –6dB angle of a line array. To determine this angle, the generic expression for the directivity function is set equal to 0.5 [3][d], i.e.

[d] The directivity function is a pressure ratio. A pressure ratio of .5 yields a sound pressure level difference of -6dB.

$$\frac{\sin u}{u} = 0.5.$$

Solving numerically, u=1.895. Since

$$u = \frac{\pi l}{\lambda}\sin\alpha$$

the quarter power angle of a uniform line array is given by

$$\begin{aligned}\theta_{-6dB} &= 2\alpha \\ &= 2\sin^{-1}\frac{1.895\lambda}{\pi l}. \\ &= 2\sin^{-1}\frac{.6\lambda}{l}\end{aligned}$$

This quarter power angle as a function of l/λ is shown in Figure 3.

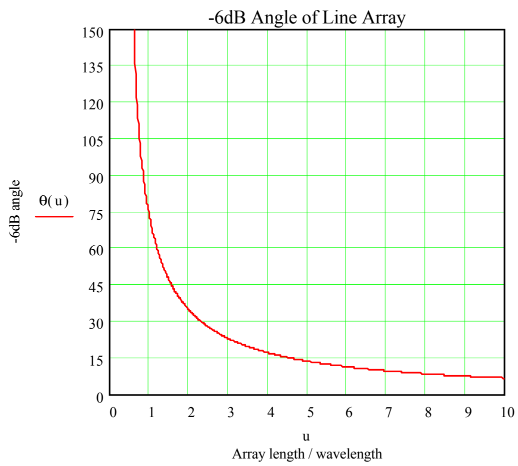

Figure 3: Quarter-power angle of a uniform line array.

At small l/λ the quarter power angle is large. At large l/λ, i.e. tall arrays and high frequency, the quarter power angle is small. For small angles[e], where $\sin(u) \approx u$, the line array directivity equation can be written as

$$\theta_{-6dB} = \frac{1.2\lambda}{l} \tag{3}$$

where θ is in radians. Expressing θ in degrees we have:

$$\theta_{-6dB} = 68.8\frac{\lambda}{l} \text{ (degrees).}$$

In some cases, it is convenient to use frequency rather than a ratio of wavelength and array length. Rewriting, the line array quarter power angle equation then becomes (approximately)

$$\theta_{-6dB} \approx \frac{24{,}000}{fl} \quad (l \text{ in meters}, f \text{ in Hz})$$

or

$$\theta_{-6dB} \approx \frac{78{,}000}{fl} \quad (l \text{ in feet}, f \text{ in Hz}).$$

The *directivity response* of an array is a plot of the quarter power angle versus frequency. The directivity response of a uniform line array, for various lengths, is shown in Figure 4.

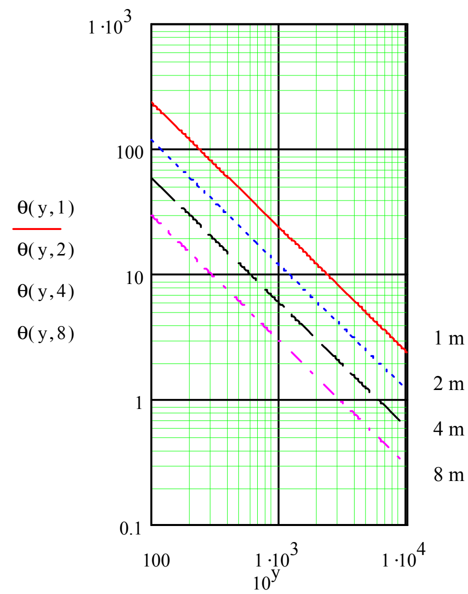

Figure 4: Directivity response of a uniform line source 1, 2, 4, and 8 meters long.

Figure 4 shows that the quarter-power angle of large arrays is quite narrow at high frequency. For instance, at 10kHz a 4-meter high array has a –6dB angle of 0.6 degrees. Arrays of this size are not uncommon in practice. Clearly, however, a sound system designer must use caution when using such an array. They are applicable only in very long throw applications and must be aligned very precisely.

[e] The small angle approximation holds for angles less than about 30 degrees. Note that $\sin(\pi/6) = 0.5000$ and $\pi/6 = 0.5235$ so that the error is less than 5%.

TAPERED LINE ARRAYS

The quarter-power angle of a line source can be increased by adjusting the amplitude along its length. A tapered line array produces a wider central lobe than a uniform array of the same length. It has a maximum amplitude at the center of the source that tapers to zero at either end. For an arbitrary maximum amplitude A, the amplitude function is given as

$$A(x) = \frac{2Ax}{l} + A \qquad \text{for } \frac{-l}{2} \le x \le 0$$
$$A(x) = \frac{-2Ax}{l} + A \qquad \text{for } 0 \le x \le \frac{l}{2}$$

and the phase function is zero. This amplitude function *A(x)* is shown in Figure 5.

The directivity function of a tapered line source is obtained by substituting this amplitude function into Equation 1:

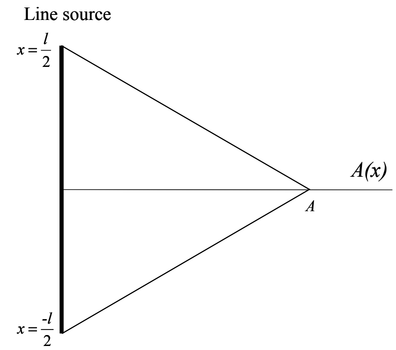

Figure 5: Graphical representation of the amplitude function of a tapered line array.

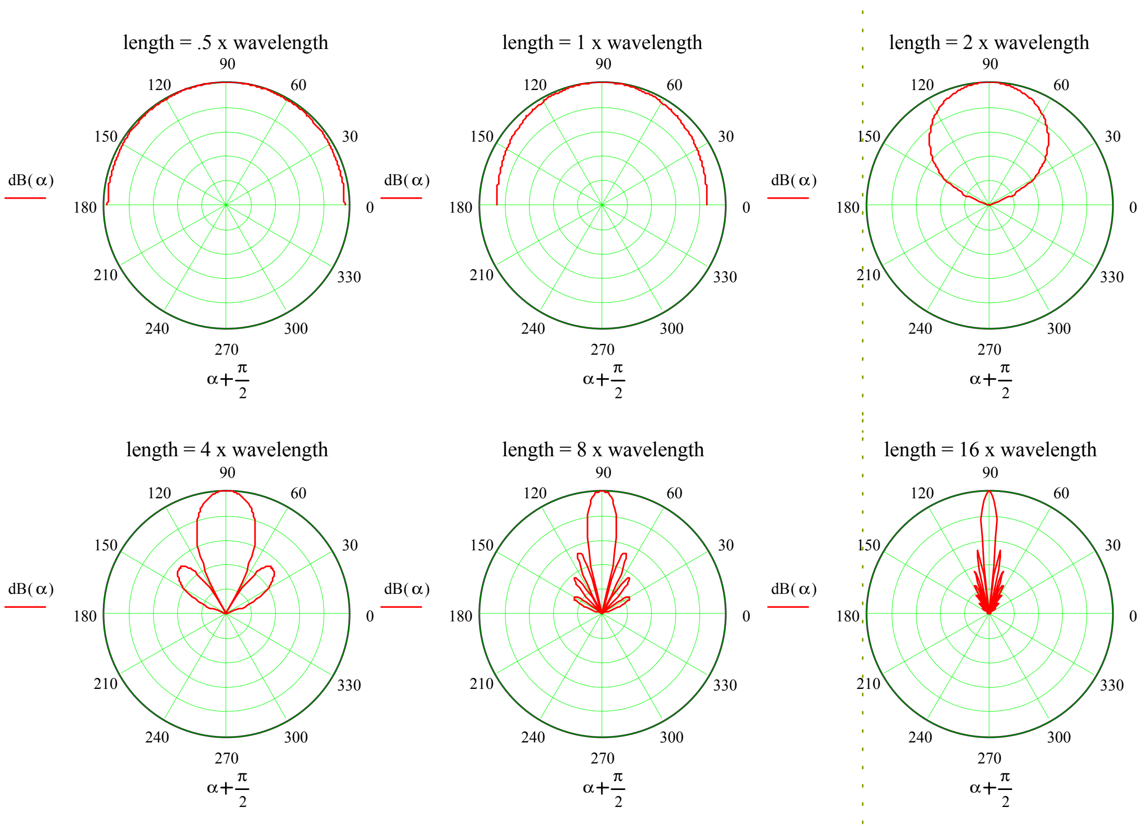

Figure 6: Polar response curves of tapered array.

$$R(\alpha)=\frac{\int_{\frac{-l}{2}}^{0} A(1+\tfrac{2x}{l})e^{-jkx\sin\alpha}dx}{\int_{\frac{-l}{2}}^{0} A(1+\tfrac{2x}{l})dx}$$

$$+\frac{\int_{0}^{\frac{l}{2}} A(1-\tfrac{2x}{l})e^{-jkx\sin\alpha}dx}{\int_{0}^{\frac{l}{2}} A(1-\tfrac{2x}{l})dx}$$

$$=\frac{\sin^2(\frac{kl}{4}\sin\alpha)}{\left(\frac{kl}{4}\sin\alpha\right)^2}\ .$$

The polar response curves of a tapered array are shown in Figure 6. For any given ratio of length and wavelength, the polars are wider than those obtained for a uniform line array. Furthermore, the side lobe structure is reduced.

The generic form of the directivity function for a tapered array is $\sin^2(u)/u^2$ where $u=(kl/4)\sin\alpha$. This always produces a wider angle than the $\sin(u)/u$ form obtained for the uniform array. The quarter-power angle for the tapered array can be determined using the earlier methodology, setting the directivity function equal to 0.5. Solving numerically,

$$\frac{\sin^2 u}{u^2}=0.5$$

yields u=1.393. Substituting for u, the quarter-power angle for a tapered array is

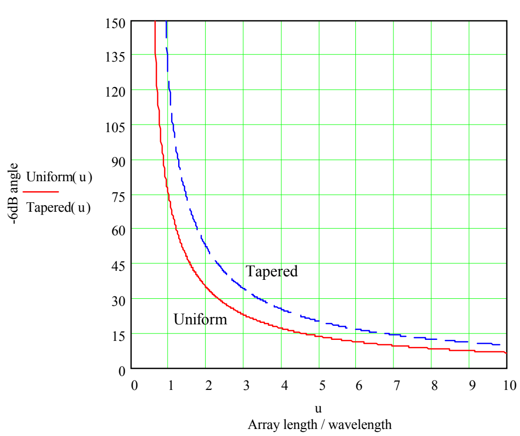

Figure 7: Quarter-power angle function for uniform and tapered line arrays.

$$\theta_{-6dB}=2\alpha$$
$$=2\sin^{-1}\frac{(1.393)(4)}{kl}$$
$$=2\sin^{-1}\frac{.9\lambda}{l}$$

Finally, using a small angle assumption as before, the quarter-power angle of a tapered array is

$$\theta_{-6dB}=\frac{1.8\lambda}{l}\ .$$

Comparing this result to Equation 3 shows that tapering the amplitude produces a quarter-power angle 50% wider than the uniform array[f]. Figure 7 provides a comparison of tapered and uniform line array quarter-power angles.

DISTANCE TO THE FAR FIELD OF A LINE ARRAY

The analysis of line arrays in the previous sections relies on a far field assumption, that is, that the distance to the point of observation **P** is large compared to the length of the array. The far field is characterized by sound pressure level decreasing at 6dB for every doubling of distance. In the near field the sound pressure level undulates and decreases nominally at 3dB per doubling of distance. The transition point can be estimated if we set as a criterion [4,5] that the far field is reached when the distance to **P** from the center point of a line array is within a quarter-wavelength of the distance to **P** from the endpoint of the array. Referring to Figure 8, the far field is obtained when

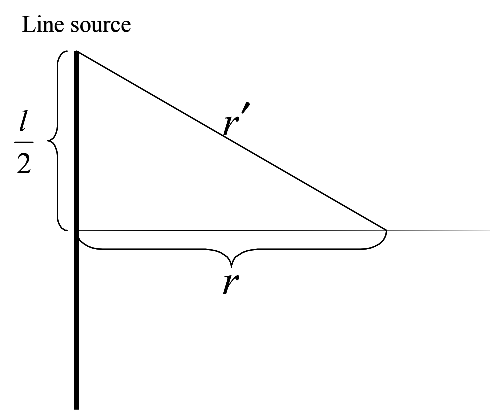

Figure 8: Geometric construction for far field distance.

$$r=r'-\frac{\lambda}{4}$$

[f] It is interesting to note that, in theory, this wider angle is obtained independent of the maximum amplitude A.

where r is the distance to P from the center point and r' is the distance from the endpoint. Solving for r',

$$r' = \sqrt{\left(\frac{l}{2}\right)^2 + r^2}$$

where l is the length of the array. Rewriting, the distance to the far field is

$$r = \frac{l^2}{2\lambda} - \frac{\lambda}{8} \quad \text{where}: l \geq \frac{\lambda}{2}. \tag{4}$$

This shows that the distance to the far field is primarily a function of the square of the array length and inversely proportional to wavelength. For most applications, the second term can be dropped[g]. For convenience, the far field equation for a line array can also be expressed in terms of frequency rather than wavelength. Two useful forms are:

$$r \approx \frac{l^2 f}{700} \quad (l \text{ in meters}, f \text{ in Hz})$$

and

$$r \approx \frac{l^2 f}{2300} \quad (l \text{ in feet}, f \text{ in Hz})$$

ON AXIS RESPONSE OF LINE ARRAYS

The on-axis response of a line array can be obtained by rewriting Equation 1 in terms of r instead of α^η. The geometric construction is shown in Figure 9. The on-axis pressure at any r is

$$pressure(r) = \int_{-l/2}^{l/2} \frac{A(x) e^{-j(kr'(x)+\phi(x))}}{r'(x)} dx$$

where $r'(x)$ is the distance to the point of observation at r from any segment dx along the line source and given by

$$r'(x) = \sqrt{r^2 + x^2}\,.$$

For a uniform line array where $A(x) = 1$ and $\phi(x) = 0$, the pressure at r is

$$pressure(r) = \int_{'l/2}^{l/2} \frac{e^{-jk\sqrt{r^2+x^2}}}{\sqrt{r^2+x^2}} dx\,.$$

[g] For wavelengths less than $l/2$, the second term is <1/16th of the first.

[h] Notice that the limits of the integration have been changed to $-l/2$ to $l/2$. This is required to place the x-axis through the center point of the line source.

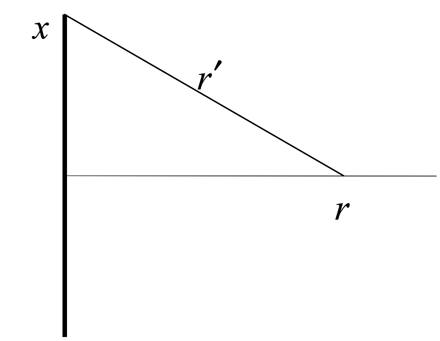

Figure 9: Geometric construction for on-axis response.

When this expression is solved numerically it shows that the distance to the far field increases with array length and frequency. Figure 10 shows the on-axis response of three arrays at 10kHz. As the length increases, the distance to the far field increases. Note that the magnitude of the on-axis response decreases at a rate of –3dB per doubling of distance in the near field and at –6dB per doubling of distance in the far field. The approximate distance at which this transition occurs was derived in the previous section and is given in Equation 4.

Figure 11 provides the on-axis response of a 4-meter long array at 100Hz, 1kHz, and 10kHz. It shows that the distance to the far field increases with frequency.

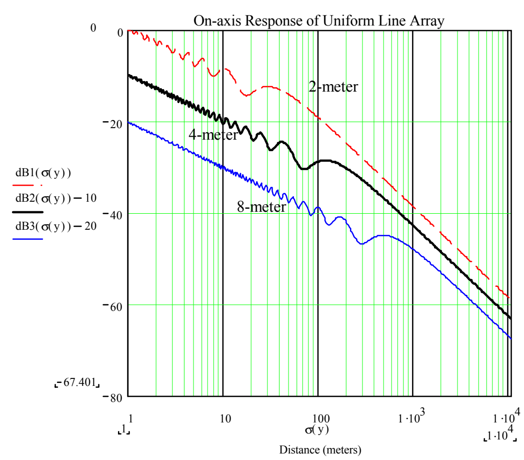

Figure 10: On-axis response of a 2, 4, and 8-meter long uniform line array at 10kHz. Note that the 4 and 8-meter responses are offset by 10 and 20dB respectively.

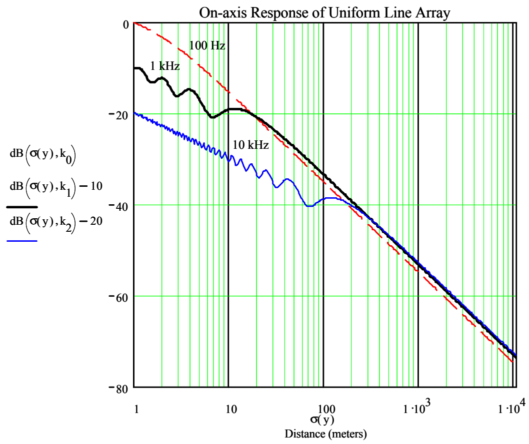

Figure 11: On-axis response of a 4-meter long uniform line array at 100Hz, 1kHz, and 10kHz. Note that the 1kHz and 10kHz responses are offset by 10 and 20dB respectively.

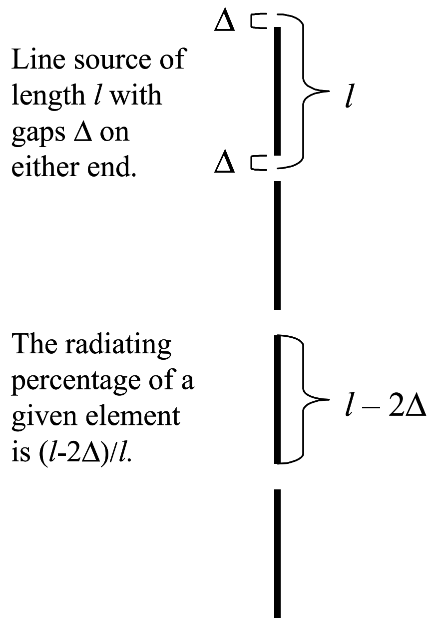

Figure 12: Line array of four elements of length *l* and non-radiating gap Δ on either end.

GAPS IN LINE ARRAYS

In previous sections we assume that line arrays have uniform amplitude and phase along their entire length. In practice, however, it may not be possible to achieve this. For instance, line arrays are often constructed of stacks of individual radiating elements such as loudspeakers or loudspeaker systems. Any gaps between elements are non-radiating and represent sections along the line array that have zero amplitude. This can be modeled by limiting the integration of Equation 2 to the radiating portions only. Referring to Figure 12, Δ is the length of the non-radiating element on either side of the radiating element.

The directivity function of a line array with n elements of length l and gaps between them is

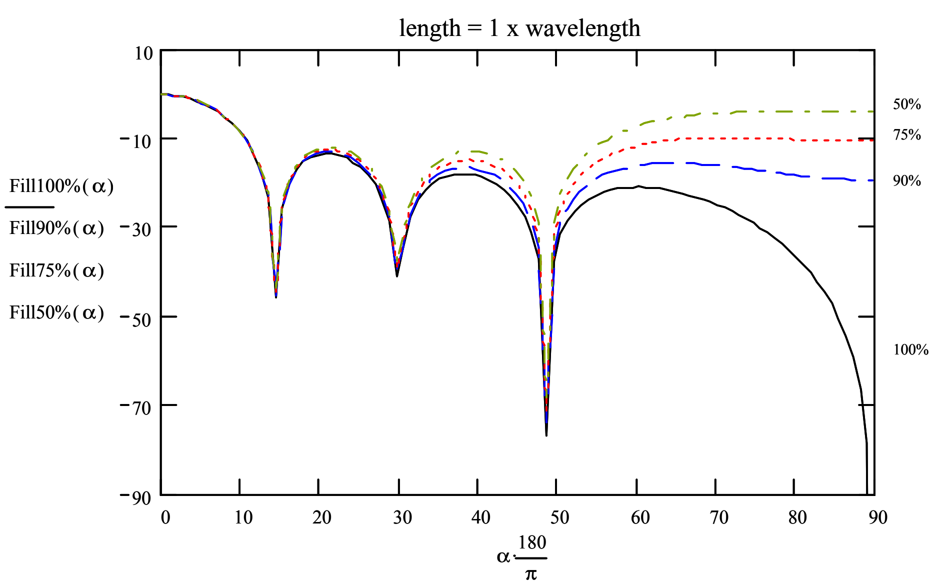

Figure 13: Directivity function of a 4-element line array with radiating percentages of 100%, 90%, 75% and 50% and where the element length is equal to the wavelength.

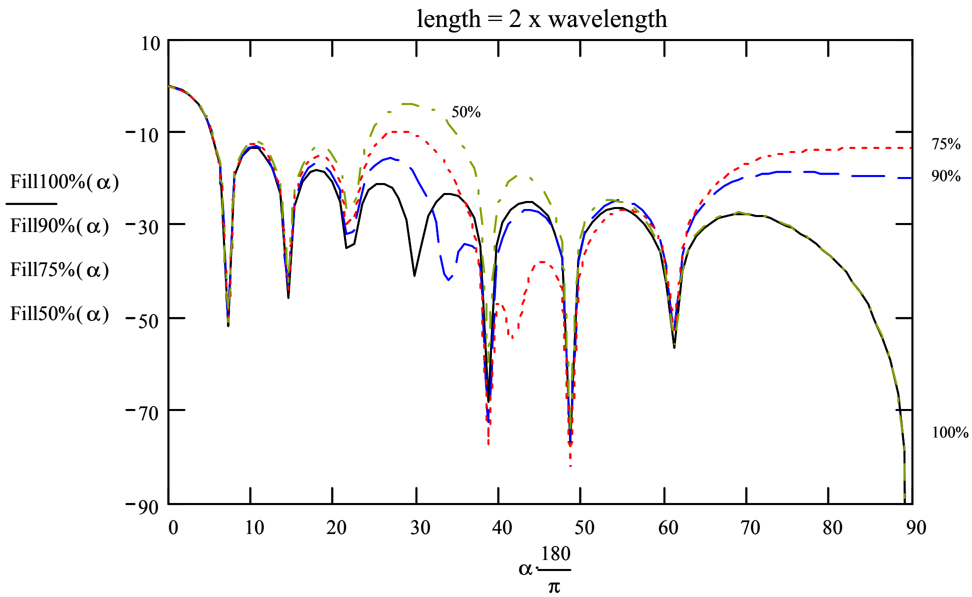

Figure 14: Directivity function of a 4-element line array with radiating percentages of 100%, 90%, 75% and 50% and where the element length is equal to two times the wavelength.

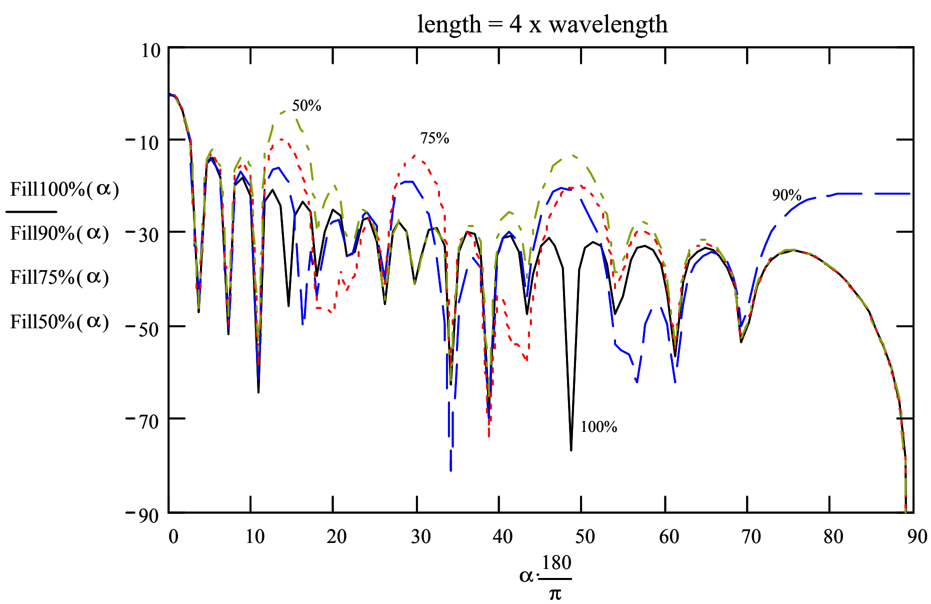

Figure 15: Directivity function of a 4-element line array with radiating percentages of 100%, 90%, 75% and 50% and where the element length is equal to four times the wavelength.

$$R(\alpha)=\sum_{1}^{n}\frac{\left|\int_{(n-1)l+\Delta}^{nl-\Delta}A(x)e^{-j(kx\sin\alpha+\phi(x))}dx\right|}{\left|\int_{(n-)l+\Delta}^{nl-\Delta}A(x)dx\right|}.$$

The gaps have very little effect on the primary lobe but change the structure of the off-axis lobes and nulls. Figures 13 through 15 show the directivity function of a four-element array as it changes with radiating percentage. Each of the four elements is 1m in length for a total length of 4 meters. The figures show the effects of changing the radiating percentage from 100% to 90%, 75% and 50% where the element length is one, two and four times the wavelength. At low frequency, where the gap length is a small fraction of the wavelength, gaps have very little effect. At high frequency, the side lobe structure changes materially with gap length. The lobes get wider and change position.

RADIAL WAVE FRONTS IN LINE ARRAYS

In practice, certain components of a line array may produce wave fronts that are curved, thereby not conforming to a strict definition of a line array. The effects that radial wave fronts have on the directivity function can be estimated by summing the radiation from a stack of curved sources. The geometric construction of a curved source is shown in Figure 16.

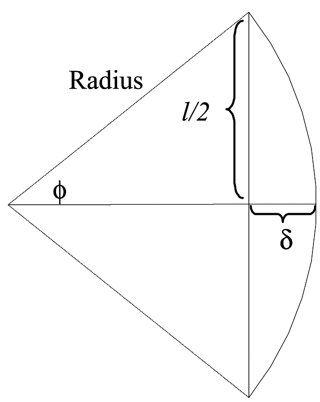

Figure 16: Geometric construction of curved

Referring to Figure 16, the directivity function of a curved (or arc) source [6] is

$$R_{\text{Arc}}(\alpha)=\left|\int_{-\phi}^{\phi}\cos\left[\frac{2\pi Radius}{\lambda}\cos(\alpha+\psi)\right]d\psi\right.$$

$$\left.+i\int_{-\phi}^{\phi}\sin\left[\frac{2\pi Radius}{\lambda}\cos(\alpha+\psi)\right]d\psi\right|$$

where *Radius* is the radius of the arc and ϕ is the half-angle of the arc. The directivity function of a stack of curved sources is obtained by applying the first product theorem[i]. In this case, the directivity function of the arc is multiplied by the directivity function of an array of simple sources. The far field directivity function for an array of n simple sources of equal amplitude and phase distributed a distance l apart along a line is given by

$$R_{\text{Points}}(\alpha)=\frac{1}{n}\left|\sum_{m=1}^{m=n}e^{-j(k(m-1)l\sin\alpha)}\right|.$$

Applying the first product theorem, the directivity function of a vertical stack of curved sources is

$$R_{\text{ArcStack}}(\alpha)=R_{\text{Arc}}(\alpha)R_{\text{Points}}(\alpha).$$

Figure 17 shows a stack of three arc sources. This stack represents an array of three loudspeaker components with non-flat wave fronts.

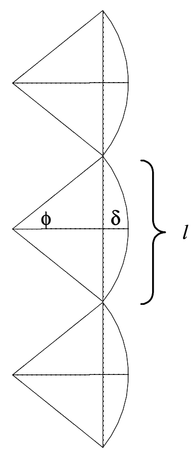

Figure 17: Stack of three curved sources of length *l*, half-angle ϕ and curvature δ.

Figures 18 through 20 show the effects of these non-flat wave fronts on the directivity function of the arrays, compared to a perfectly flat wave front. As with gaps in an array, curvature primarily produces changes in the lobe/null structure of the off-axis response. The changes increase with increasing curvature and are predominant when the curvature is a material fraction of a wavelength.

Figure 18 compares the directivity function of a pure line source of length 3l with an array of three curved sources (Figure 17) of length l, where the curvature δ of the arc is one-eighth wavelength. The directivity functions are very similar, with only small differences in the lobe/null structure. In particular, note that the nulls at approximately 18º, 35º and 60º are not as deep with the stack of curved sources.

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i The first product theorem states that the directional factor of an array of identical sources is the product of the directional factor of an array and the directional factor of a single element of the array. See Kinsler and Frey [4].

Figures 19 and 20 show the directivity functions of the pure line and the three-element array at curvatures of ¼ and ½ wavelengths. In these cases, lobes gradually replace the nulls at 18º, 35º and 60º. At ¼ wavelength, the lobe at 18º is approximately 10dB below the level of the on-axis lobe, up from approximately 20dB. This represents a practical limit to curvature that maintains, generally speaking, the directivity function of a pure line array. The response at ½ wavelength is unacceptable as the 18º lobe is here nearly equal in amplitude to the primary lobe.

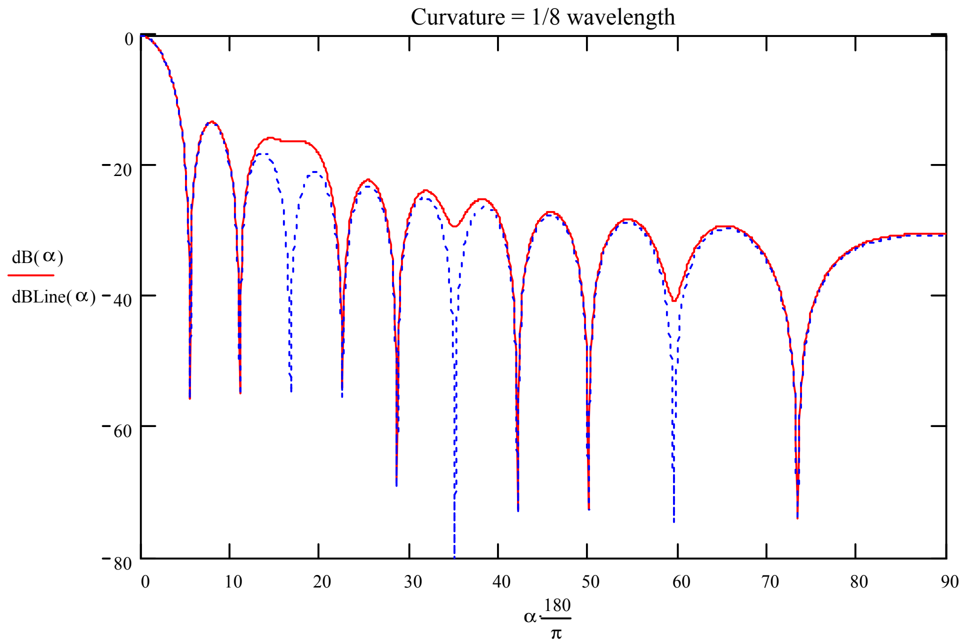

Figure 18: Comparison of directivity functions of a stack of three curved sources (solid line) and a pure line source (dotted line). The curved sources have an element length *l* of 15 cm, a total included angle ϕ of 20 degrees, and a curvature δ of 1/8 wavelength. The pure line source has a total length of 3*l*.

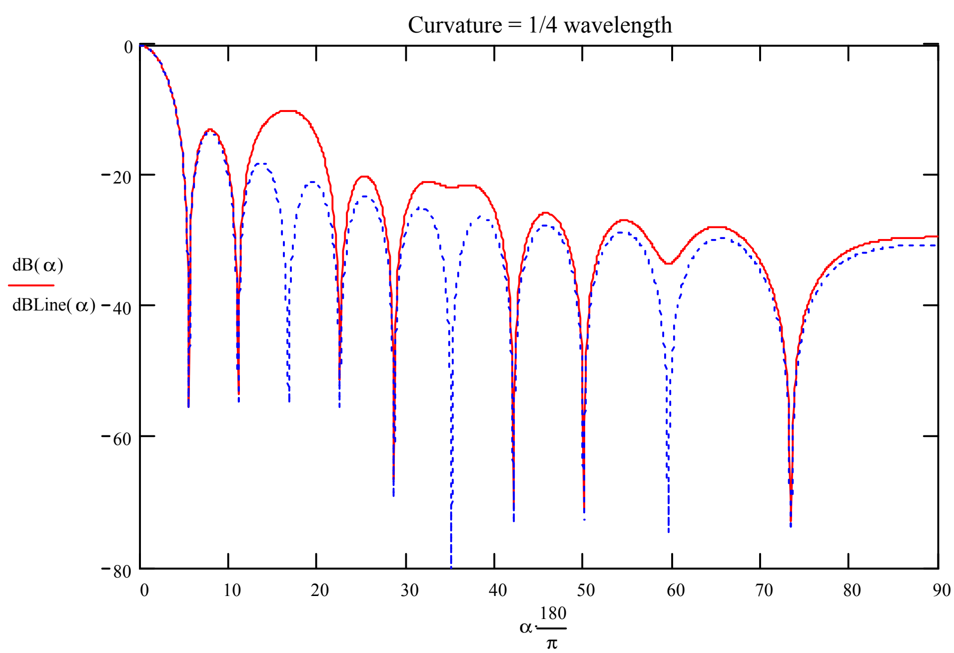

Figure 19: As above where δ is ¼ wavelength.

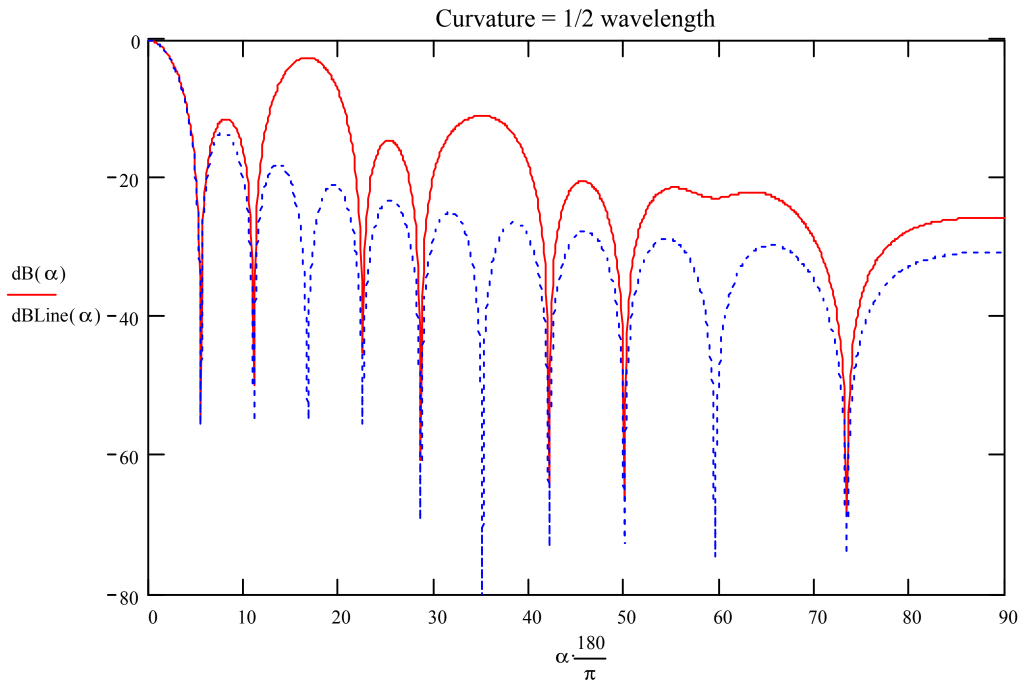

Figure 20: As above where δ is 1/2 wavelength.

This ¼ wavelength limit on curvature allows us to estimate, for a given curved source, the practical upper frequency limit for which it maintains the directivity response of a pure line array. If a source has an element length l of 15 cm and a total arc angle of 20º (ϕ = 10º), then

$$\delta = \frac{l}{2}\tan\left(\frac{\phi}{2}\right) = .66cm$$

and the upper frequency limit is

$$f = \frac{c}{4\delta} \approx 13kHz\,.$$

SUMMARY

This paper provides mathematical models and methods for analyzing the directivity of line arrays. The general form of the line array directivity function is derived and used to estimate the lobe/null structure and the quarter-power angle as functions of array length and wavelength. Since this derivation assumes a far field condition, an expression is derived for the distance to the far field. This distance corresponds to the transition point, shown in on-axis pressure response curves, where the undulating pressure response of the near field yields to the familiar inverse square law. The paper shows how the transition distance is a function of length and wavelength.

Finally, recognizing that large line arrays of loudspeakers may not be perfect line sources, the paper analyzes the effects of certain imperfections on line array directivity. These imperfections include gaps between the radiating portions of loudspeakers and curvature of the radiated wave front. Recent loudspeaker systems, designed specifically for use in line arrays, attempt to minimize these imperfections. Nonetheless, it is useful to quantify their effects on directivity response. The paper shows that gaps and non-flat wave fronts can change the off-axis response, particularly where the gap or amount of curvature is a material fraction of the wavelength.

REFERENCES

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[3] Benson, J. E., "Theory and Applications of Electrically Tapered Electro-Acoustic Arrays," IREE International Electronics Convention Digest, pp. 587-589, August 1975. *Note that Benson solved for the –3dB angle.*

[4] Kinsler and Frey, "Fundamentals of Acoustics", third edition, John Wiley and Sons, 1980. Page 188 shows a similar derivation using ½ wavelength as the criterion.

[5] Smith, D. L., Discrete-Element Line Arrays – Their Modeling and Optimization, J. Audio Eng. Soc., Vol. 45, No. 11, 1997.

[6] The reader is referred to Wolff and Malter or Olson above. They provide a point source derived expression for the directivity function of a curved source. The integral expression was derived by the author.